ON LEAST ENERGY SOLUTIONS TO A SEMILINEAR ELLIPTIC EQUATION IN A STRIP

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Dedicated to Professor L. Nirenberg on the occasion of his 85^{th} birthday, with deep admiration

ABSTRACT. We consider the following semilinear elliptic equation on a strip:

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \mathbb{R}^{N-1} \times (0, L), \\ u > 0, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial(\mathbb{R}^{N-1} \times (0, L)) \end{cases}$$

where $1 < p \leq \frac{N+2}{N-2}$. When $1 < p < \frac{N+2}{N-2}$, it is shown that there exists a unique $L_* > 0$ such that for $L \leq L_*$, the least energy solution is trivial, i.e., doesn't depend on x_N, and for $L > L_*$, the least energy solution is nontrivial. When $N \geq 4, p = \frac{N+2}{N-2}$, it is shown that there are two numbers $L_* < L_{**}$ such that the least energy solution is trivial when $L \leq L_*$, the least energy solution is nontrivial when $L \in (L_*, L_{**}]$, and the least energy solution does not exist when $L > L_{**}$. A connection with Delaunay surfaces in CMC theory is also made.

1. INTRODUCTION

In this paper, we consider the following semilinear elliptic equation on a strip

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \mathbb{R}^{N-1} \times (0, L), \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial(\mathbb{R}^{N-1} \times (0, L)), \\ u > 0, \ u \in H^1(\mathbb{R}^{N-1} \times (0, L)). \end{cases} \tag{1.1}$$

Here we assume that

$$N \geq 2, 1 < p \leq \frac{N+2}{N-2} \text{ if } N \geq 3, \text{and } 1 < p < +\infty \text{ if } N = 2$$

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and ν is the outer normal derivative.

The motivation of this study stems of the work of Dancer on new solutions to for the following simple superlinear problem

$$\Delta u - u + u^p = 0 \text{ in } \mathbb{R}^N, u > 0, p > 1. \tag{1.2}$$

If $u(x) \to 0$ as $|x| \to +\infty$, then the classical work of Gidas-Ni-Nirenberg [12] shows that u must be *radially symmetric* with respect to one point and thus (1.2) is reduced to an ODE. On the other hand, there are less known results on solutions to (1.2) which do not decay in all directions. Dancer [6] first constructed solutions to (1.2) that are periodic in one direction and decays in all the other directions, via local bifurcation arguments. They form a one-parameter family of solutions which are periodic in the z variable and originate from the decaying solutions of (1.2) in $\mathbb{R}^{N-1}$. We briefly outline Dancer's idea: Let $T > 0$ be the period and consider

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \mathbb{R}^N, u > 0, \\ u(x^{'}, x_N + T) = u(x^{'}, x_N), u(x^{'}, x_N) \to 0 \text{ as } |x^{'}| \to +\infty \end{cases} \tag{1.3}$$

where we denote $x^{'} = (x_1, ..., x_{N-1})$. Dancer then used T as the bifurcation parameter and found a critical value T_1 such that for $T = T_1$, the linearized problem at the lower dimensional decaying solutions has an eigenvalue zero with eigenfunctions decaying in $x^{'}$. Then using the Crandall-Rabinowitz bifurcation theory, near T_1, a new solution (different from lower dimensional solution) bifurcates.

In [10], these periodic solutions are called *Dancer's solutions* and they are the building blocks for more complicated "$2k$-ends" solutions. In [22], Dancer's solutions are also used to build *three ends* solutions to the problem in entire space. In fact, geometrically, Dancer's solutions corresponds to the so-called Delaunay solution in CMC theory [8]. We will comment on this later. Therefore it becomes natural to study the solution structure of (1.1).

Problem (1.1) also arises naturally in the study of some nonlinear elliptic equations in an expanding annuli:

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } B_{R+L} \backslash B_R, \\ u > 0, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial(B_{R+L} \backslash B_R), \end{cases} \tag{1.4}$$

where $R \to +\infty$ and L is fixed. The limiting equation of (1.4) as $R \to +\infty$ becomes (1.1).

We note that the corresponding expanding annuli Dirichlet problem

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } B_{R+L} \backslash B_R, \\ u > 0, u = 0 \text{ on } \partial(B_{R+L} \backslash B_R), \end{cases} \tag{1.5}$$

has been studied by many authors, see [4], [5], [7], [17], [18], [19], [23] and the references therein. We are not aware of any study on (1.4). Note that (1.4) is different from (1.5). Indeed, (1.4) admits solutions that are nonzero and

lower dimensional, i.e. don't depend on x_N-direction. The issue, therefore, is to understand when and how solutions that are not lower dimensional exist.

By suitable scaling (1.1) becomes

$$\begin{cases} \Delta u - L^2 u + u^p = 0 \text{ in } \Sigma := \mathbb{R}^{N-1} \times (0,1), \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma,\ u > 0,\ u \in H^1(\Sigma). \end{cases} \tag{1.6}$$

Here L is the parameter. In this paper, we consider the existence or non-existence as well as nature of *least energy solutions*. More precisely, let

$$c(L) := \inf_{u \in H^1(\Sigma), u \not\equiv 0} \frac{\int_\Sigma (|\nabla u|^2 + L^2 u^2)}{\left(\int_\Sigma u^{p+1}\right)^{\frac{2}{p+1}}}. \tag{1.7}$$

Our main concerns are:
Q1: Is $c(L)$ attained?
Q2: Is $c(L)$ attained by a nontrivial solution?
Q3: Is the least energy solution nondegenerate?

Here, a trivial solution is understood to mean that the solution does not depend on x_N. Note that for $p < \frac{N+1}{N-3}$ if $N \geq 4$ and $1 < p < +\infty$ if $N = 2, 3$, such trivial solutions of (1.6) always exist. Indeed, a ground state solution ([2]-[3]) in $\mathbb{R}^{N-1}$ yields such a "trivial solution".

Our main theorem is

Theorem 1.1. *(1) If $p < \frac{N+2}{N-2}$ when $N \geq 3$ and $p < +\infty$ when $N = 2$, then there exists a unique L_* such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution and for $L > L_*$, $c(L)$ is attained by a nontrivial solution.*

(2) There exist $L_2 \geq L_*$ such that the least energy solution is unique and nondegenerate for any $L \geq L_2$.

(3) If $N \geq 4, p = \frac{N+2}{N-2}$, then there exists two positive constants $L_* < L_{**}$ such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution; for $L \in (L_*, L_{**}]$, $c(L)$ is attained by a nontrivial solution; for $L > L_{**}$, $c(L)$ is not attained.

Remark: The number L^* can be computed as follows: Let w_0 be the unique ground state solution in $\mathbb{R}^{N-1}$

$$\Delta w_0 - w_0 + w_0^p = 0,\ w_0 = w_0(|x'|) > 0, w_0 \in H^1(\mathbb{R}^{N-1}). \tag{1.8}$$

(See [2], [15].) Let λ_1 be the unique principal eigenvalue of

$$\Delta\phi - \phi + p w_0^{p-1}\phi = \lambda_1 \phi, \phi \in H^1(\mathbb{R}^{N-1}). \tag{1.9}$$

Then we have

$$L^* = \frac{\pi}{\sqrt{\lambda_1}}. \tag{1.10}$$

When $N = 2$, we can compute explicitly (see [9])

$$\lambda_1 = \frac{(p-1)(p+3)}{4}. \tag{1.11}$$

In fact, we can say more about the properties of the minimizers and the asymptotic behaviors of $c(L)$ as $L \to 0$ or $L \to +\infty$. The asymptotic behavior of the least energy solution when $L \to L_*$ is given in the appendix.

Even though Theorem 1.1 is a purely PDE result, this result has a striking analogy in the theory of constant mean curvature (CMC) surface in $\mathbb{R}^3$.

CMC surfaces in $\mathbb{R}^3$ are equilibria for the area functional subjected to an enclosed volume constraint. It arises in many physical and variational problems. Over the past two decades a great deal of progress was achieved in understanding complete CMC surfaces and their *moduli* spaces. Spheres (zero end) and round cylinders are the first examples of CMC surfaces. (See Alexandrov's [1].) Properly embedded CMC surfaces with nonzero mean curvature were classified by Delaunay [8]. These are CMC rotation surfaces, called *unduloids* (having genus zero and two ends). These surfaces are derived from two 1-parameter families: one of the family being unduloids with neck radius $\tau \in (0, \frac{1}{2}]$ and the other being a family of non-embedded surface called *nodoids* that can be parameterized by the neck radius $\tau \in (0, \infty)$.

In very much an analogous way, our solutions in Theorem 1.1 are parameterized by the length L. When $L \to +\infty$, these solutions become spikes at the center and correspond to the Delaunay surface that are obtained when $\tau \to 0$. On the other hand, when $L \to L_*$ our solution corresponds to Delaunay solution when $\tau \to \frac{1}{2}$. One good way to think of this analogy is the level sets of u. (See [10] for more explanations.) In [10], del Pino-Kowalczyk-Pacard-Wei used the least energy solution near L^* and Toda systems to build more complicated even-ended solutions of (1.2) in $\mathbb{R}^2$, while in [22], Malchiodi used the least energy solution near $+\infty$ to build $Y-$shaped solutions.

We conjecture that the least energy solution form a continuous family as L goes from L_* to $+\infty$.

After the paper was completed, we learned from Prof. M. Esteban that problem (1.6) is also related to the study of Cafferalli-Kohn-Nirenberg inequality and it is studied in the work of Dolbeault, Esteban, Loss and Tarantello [11].

This paper is organized as follows: we prove (1), (2) and (3) of Theorem 1.1 in Sections 2,3 and 4 respectively. In Appendix A, we prove some technical estimates used in Section 4 while in Appendix B we study the asymptotic behavior of least energy solutions when $L \to L_*$.

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2. The subcritical case: Proof of (1) of Theorem 1.1

In this section, we study (1.6) for the subcritical case, i.e., $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ when $N = 2$. We begin with

Lemma 2.1. *For any $L_0 > 0$, there exists a constant C, independent of $L \leq L_0$, such that for any solution of (1.6) we have*

$$u \leq C. \tag{2.1}$$

Proof. This follows from standard blowing-up argument. For the sake of completeness, we include a short proof here. Suppose (2.1) were not true. Then, there would exist a sequence of functions u_i and $L_i \leq L_0$ such that $M_i := \sup_{x\in\Sigma} u_i(x) \to +\infty$. Without loss of generality, we may assume that $M_i = u_i(x_i), x_i = (0', x_{i,N})$. Indeed, for fixed i, there exists a sequence of points x_i^k such that $u_i(x_i^k) \to \sup_{\bar{\Sigma}} u_i$ as $k \to +\infty$. Then let $x_i^k = ((x_i^k)', x_{i,N}^k)$ and define $u_i^k(x) = u_i(x' + (x_i^k)', x_N)$. Then, using standard elliptic estimates, one can strike out a sequence u_i^k converging to u_i^∞ as $k \to +\infty$. We may then replace u_i by u_i^∞ (which we call u_i again). Then, u_i is a solution of (1.6) and there exists $x_i = (0', x_N^i)$ such that $u_i(x_i) = \max_{\bar{\Sigma}} u_i := M_i$.

Then we perform a classical blow up analysis. Set

$$\epsilon_i = M_i^{-\frac{p-1}{2}}, v_i(y) = \epsilon_i^{\frac{2}{p-1}} u_i(x_i + \epsilon_i y). \tag{2.2}$$

Then it is easy to see that $v_i(y)$ satisfies

$$\Delta v_i - L_i^2 \epsilon_i^2 v_i + v_i^p = 0 \text{ in } \Sigma_i = \mathbb{R}^{N-1} \times (-\frac{x_{i,N}}{\epsilon_i}, \frac{L_i - x_{i,N}}{\epsilon_i}).$$

Now we extend v_i to $\mathbb{R}^N$ be periodic extension. We still denote the periodic extension as v_i. Then, up to extraction of a subsequence, $v_i(y) \to v_0(y)$ in $C^2_{loc}(\mathbb{R}^N)$, and v_0 satisfies the equation

$$\Delta v_0 + v_0^p = 0 \text{ in } \mathbb{R}^N, v_0(0) = 1.$$

However, this is clearly impossible by the result of Gidas-Spruck [13]. □

As a corollary, we have

Corollary 2.2. *There exists a $L_* > 0$ such for $L < L_*$, $c(L)$ is achieved only by trivial solutions.*

Proof. Certainly by Sobolev embedding theorem and Steiner's symmetrization, a minimizer which is symmetric in x' to $c(L)$ exists. We call it u_L. Now consider the function $\phi = \frac{\partial u_L}{\partial x_N}$. Then ϕ satisfies

$$\Delta\phi - L^2\phi + p u_L^{p-1}\phi = 0 \text{ in } \Sigma,\ \phi = 0 \text{ on } \partial\Sigma. \tag{2.3}$$

As $L \to 0$, since u_L is uniformly bounded by Lemma 2.1 and by the same argument as in the proof of Lemma 2.1, we conclude that $\sup_{x\in\Sigma} u_L \to 0$.

On the other hand, since $\phi \in H_0^1(\Sigma)$, by Poincare's inequality

$$C\int_\Sigma |\phi|^2 \le \int_\Sigma |\nabla \phi|^2 \tag{2.4}$$

for some positive constant C.

Multiplying (2.3) by ϕ, we obtain using (2.4)

$$p\int_\Sigma u_L^{p-1}\phi^2 = \int_\Sigma |\nabla\phi|^2 + L^2\int_\Sigma \phi^2 \ge C\int_\Sigma \phi^2. \tag{2.5}$$

This is impossible if $\phi \not\equiv 0$, for L small enough since $\sup_{x\in\Sigma} u_L \to 0$. Therefore for L small enough, $\frac{\partial u_L}{\partial x_N} \equiv 0$ and u_L is independent of x_N. $\square$

Let us denote by $c^*(L)$ the energy level of the trivial solutions, i.e., solutions depending on $x^{'}$ only. By a simple computation, we see that

$$c^*(L) = \gamma_0 L^{2-\frac{(N-1)(p-1)}{p+1}}. \tag{2.6}$$

where γ_0 is generic constant (independent of L). Certainly $c(L) \le c^*(L)$.

Lemma 2.3. *As $L \to +\infty$, $L^{-\frac{2}{p-1}} u_L(L^{-1}y) \to w(y)$ where $w(y)$ is the unique solution of*

$$\begin{cases} \Delta w - w + w^p = 0 \ \textit{in } \mathbb{R}^N, \\ w > 0 \ \textit{in } \mathbb{R}^N, w(0) = \max_{y\in\mathbb{R}^N} w(y), w(y) \to 0 \ \textit{as } |y| \to +\infty \end{cases} \tag{2.7}$$

In fact, u_L has only one local maximum point P_L on $\partial\Sigma$.

Proof. By Schwarz spherical rearrangement with respect to $x^{'}$ and Steiner monotone increasing rearrangement in x_N, after a shift of the origin and a change x_N to $-x_N$ if needed, we see that u_L is radially symmetric in $x^{'}$ and monotone increasing in x_N. Therefore, there exists a unique point $P_0 = (0^{'}, 1)$ where u_L achieves a maximum. Let $v_L(y) = L^{-\frac{2}{p-1}} u_L(P_0 + L^{-1}y)$. Then v_L satisfies

$$\Delta v_L - v_L + v_L^p = 0 \ \text{ in } \Sigma_L, v_L \in H^1(\Sigma_L)$$

where $\Sigma_L = \mathbb{R}^{N-1} \times (-L, 0)$. Simple computations show that

$$c(L) = (\int_\Sigma u_L^{p+1})^{\frac{p-1}{p+1}} = L^{2-\frac{N(p-1)}{p+1}} (\int_{\Sigma_L} v_L^{p+1})^{\frac{p-1}{p+1}}. \tag{2.8}$$

Since w decays exponentially, we can use a cut-off of w to be a test function and derive that

$$c(L) \le (\frac{1}{2}\int_{\mathbb{R}^N} w^{p+1} + O(L^{-1}))^{\frac{p-1}{p+1}} L^{2-\frac{N(p-1)}{p+1}} \tag{2.9}$$

for $L >> 1$. We conclude that $\int_{\Sigma_L} v_L^{p+1}$ is bounded and hence the $H^1(\Sigma_L)$ norm of v_L is bounded. Consequently, standard arguments show that as $L \to +\infty$, $v_L(y) \to w(y)$. $\square$

The following lemma gives part of (1) of Theorem 1.1.

Lemma 2.4. *Let L_0 be such that $c(L_0) < c^*(L_0)$. Then for any $L > L_0$, it holds that $c(L) < c^*(L)$.*

Proof. Since $c(L_0) < c^*(L_0)$, $c(L_0)$ is attained by a nontrivial solution, which we call $v_0(x^{'}, x_N)$. Thus, v_0 satisfies

$$\Delta v_0 - L_0^2 v_0 + v_0^p = 0 \text{ in } \Sigma. \tag{2.10}$$

Note that since v_0 is nontrivial, we have

$$\int_\Sigma v_{0,x_N}^2 > 0. \tag{2.11}$$

Now let us consider the following transformation:

$$u(x^{'}, x_N) = \lambda^{\frac{2}{p-1}} v_0(\lambda x^{'}, x_N), \text{ where } \lambda = \frac{L}{L_0}. \tag{2.12}$$

A simple computation shows that $u(x)$ satisfies

$$\Delta u - L^2 u + u^p + (\lambda^2 - 1) u_{x_N x_N} = 0. \tag{2.13}$$

Hence

$$\int_\Sigma (|\nabla u|^2 + L^2 u^2) = \int_\Sigma u^{p+1} - (\lambda^2 - 1) \int_\Sigma |u_{x_N}|^2. \tag{2.14}$$

Now, since $\lambda > 1$, we derive the following sequence of inequalities

$$\begin{aligned} c(L) &\leq \frac{\int_\Sigma (|\nabla u|^2 + L^2 u^2)}{(\int_\Sigma u^{p+1})^{\frac{2}{p+1}}} \\ &< (\int_\Sigma u^{p+1})^{\frac{p-1}{p+1}} = \lambda^{2 - \frac{(N-1)(p-1)}{p+1}} (\int_\Sigma v_0^{p+1})^{\frac{p-1}{p+1}} \\ &< \lambda^{2 - \frac{(N-1)(p-1)}{p+1}} c^*(L_0) = c^*(L). \end{aligned}$$

□

The following inequality may be of independent interest.

Lemma 2.5. *Let u_L be a least energy solution to $c(L)$. Then it holds*

$$\int_\Sigma [(|\nabla \varphi|^2 + L^2 \varphi^2) - p u_L^{p-1} \varphi^2] + (p-1) \frac{(\int_\Sigma u_L^p \varphi)^2}{\int_\Sigma u_L^{p+1}} \geq 0, \forall \varphi \in H^1(\Sigma). \tag{2.15}$$

Proof. This follows from the variational characterizations of u_L. In fact, let

$$Q[u] = \frac{\int_\Sigma (|\nabla u|^2 + L^2 u^2)}{(\int_\Sigma u^{p+1})^{\frac{2}{p+1}}}$$

and $\rho(t) = Q[u_L + t\varphi]$ for any $\varphi \in H^1(\Sigma)$. Since $\rho(t) \geq \rho(0)$ for any t, $\rho^{'}(0) = 0, \rho^{''}(0) \geq 0$. Note that

$$
\begin{aligned}
&\int_\Sigma (|\nabla(u_L + t\varphi)|^2 + L^2(u_L + t\varphi)^2) \\
= \quad &\int_\Sigma (|\nabla u_L|^2 + L^2 u_L^2) + 2t\int_\Sigma (\nabla u_L \nabla\varphi + L^2 u_L \varphi) + t^2 \int_\Sigma (|\nabla\varphi|^2 + L^2\varphi^2) \\
= \quad &\int_\Sigma u_L^{p+1} + 2t\int_\Sigma u_L^p \varphi + t^2 \int_\Sigma (|\nabla\varphi|^2 + L^2\varphi^2) \\
&\int_\Sigma (u_L + t\varphi)^{p+1} = \int_\Sigma u_L^{p+1} + (p+1)t\int_\Sigma u_L^p\varphi + \frac{(p+1)p}{2}\int_\Sigma u_L^{p-1}\varphi^2 + \mathcal{O}(t^2)
\end{aligned}
$$

Then that $\rho^{''}(0) \geq 0$ is equivalent to

$$
\int_\Sigma [(|\nabla\varphi|^2 + L^2\varphi^2) - pu_L^{p-1}\varphi^2] + (p-1)\frac{(\int_\Sigma u_L^p\varphi)^2}{\int_\Sigma u_L^{p+1}} \geq 0. \tag{2.16}
$$

□

The next result is a corollary of inequality (2.15).

Lemma 2.6. *Let u_L be a least energy solution of $c(L)$ and $\lambda_2(u_L)$ be the second eigenvalue of*

$$
\Delta\phi - L^2\phi + pu_L^{p-1}\phi + \lambda\phi = 0 \ \textit{in}\ \Sigma, \ \frac{\partial\phi}{\partial\nu} = 0 \ \textit{on}\ \partial\Sigma. \tag{2.17}
$$

Then, necessarily

$$
\lambda_2(u_L) \geq 0. \tag{2.18}
$$

Proof. Recall that by the Courant-Fisher-Weyl formula, one has

$$
\lambda_2 = \max_{dim(V)=1, V\in H^1(\Sigma)} \inf_{\varphi\perp V} \int_\Sigma [(|\nabla\varphi|^2 + L^2\varphi^2) - pu_L^{p-1}\varphi^2]. \tag{2.19}
$$

Then by choosing $V = \text{span}\{u_L^p\}$ in (2.19) and using (2.15), we derive that $\lambda_2 \geq 0$. □

Completion of proof of (1) of Theorem 1.1: Let

$$
L_* = \sup\{L | c(l) = c^*(l) \text{ for } l \in (0, L)\}.
$$

By Corollary 2.2, we see that $0 < L_*$. Now from Lemma 2.3, it follows that $L_* < +\infty$. Indeed, for L large, we have by (2.9) and Lemma 2.3, $c(L) \sim L^{2-\frac{N(p-1)}{p+1}}$, while $c^*(L) \sim L^{2-\frac{(N-1)(p-1)}{p+1}}$. Certainly, for $l \leq L_*$, $c(l) = c^*(l)$ and u_l is trivial. By Lemma 2.4, $c(L) < c^*(L)$ for $L > L_*$.

We now claim that $L^* = \frac{\pi}{\sqrt{\lambda_1}}$. In fact, by separation of variables, the second eigenvalue of the following eigenvalue problem

$$\Delta \phi - L^2 \phi + p w_0^{p-1} \phi + \lambda \phi = 0 \text{ in } \Sigma, \ \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \Sigma \tag{2.20}$$

is

$$\lambda_2 = \pi^2 - L^2 \lambda_1 \geq 0. \tag{2.21}$$

By Lemma 2.6, $L^* \leq \frac{\pi}{\sqrt{\lambda_1}}$. On the other hand, if $L^* > \frac{\pi}{\sqrt{\lambda_1}}$, then by Lemma 2.4, w_0 is a minimizer of c_L when L is close to $\frac{\pi}{\sqrt{\lambda_1}}$. But it is easy to see that w_0 loses its stability exactly at $L = \frac{\pi}{\sqrt{\lambda_1}}$ by (2.21).

3. The subcritical Case: the proof of (2) of Theorem 1.1

In this section, we prove the uniqueness and nondegeneracy of the least energy solutions for $L \to +\infty$.

First we recall

Lemma 3.1. *Let w be the least energy solution of*

$$\Delta w - w + w^p = 0, w > 0, w \in H^1(\mathbb{R}^N). \tag{3.1}$$

Then w is nondegenerate, i.e.,

$$Ker\ (\Delta - 1 + p w^{p-1}) = span\ \{\frac{\partial w}{\partial y_1}, ..., \frac{\partial w}{\partial y_N}\} \tag{3.2}$$

Proof. The result is well-known. By the clasical result of Gidas-Ni-Nirenberg [12], w is radially symmetric. The nondegeneracy follows from the uniqueness result of Kwong [15]. See Lemma A.3 of [25]. Here we include a short and self-contained new proof of nondegeneracy using only property of least energy. This part is of independent interest, but it is restricted to the power nonlinearity.

Let $w = w(r)$ be a radial least energy solution of (2.7). By the same proof as in Lemma 2.6, $\lambda_{2,r}(w) \geq 0$, where $\lambda_{2,r}$ denotes the second eigenvalue in the radial class. It remains to show that $\lambda_{2,r}(w) > 0$. Suppose $\lambda_{2,r} = 0$ and let $\phi(r)$ be the corresponding eigenfunction, i.e.

$$\Delta \phi - \phi + p w^{p-1} \phi = 0, \phi = \phi(r) \in H^1(\mathbb{R}^N). \tag{3.3}$$

Then the characterization of the second eigenfunction implies that ϕ changes sign once. So we may assume that $\phi < 0$ for $r < r_0$ and $\phi > 0$ for $r > r_0$. Now as in Kwong-Zhang [16] we consider the function

$$\eta(r) = r w^{'} - \beta w. \tag{3.4}$$

Then η satisfies

$$\Delta \eta - \eta + p w^{p-1} \eta = 2w - (2 + \beta(p-1)) w^p. \tag{3.5}$$

We choose β such that $1=(1+\frac{\beta(p-1)}{2})w^{p-1}(r_0)$, hence $2w-(2+\beta(p-1))w^p<0$ for $r<r_0$ and $2w-(2+\beta(p-1))w^p>0$ for $r>r_0$.

Multiplying (3.3) by η and (3.5) by η, we arrive at

$$\int_{\mathbb{R}^N}\phi(2w-(2+\beta(p-1))w^p)=0 \tag{3.6}$$

which is impossible by the property of ϕ. Thus $\phi\equiv 0$ and this completes the proof. □

Let us now prove the nondegeneracy of the least energy solution when L is large. By the rescaling $w_L=L^{-\frac{2}{p-1}}u_L((0,1)+L^{-1}y)$, it is enough to show that the only solutions to

$$\Delta\phi-\phi+pw_L^{p-1}\phi=0, \phi\in H^1(\Sigma_L) \tag{3.7}$$

are $\frac{\partial w_L}{\partial y_j}, j=1,...,N-1$. Here $\Sigma_L=\mathbb{R}^{N-1}\times(-L,0)$. We may assume that

$$\int_{\Sigma_L}\phi\frac{\partial w_L}{\partial y_j}=0, j=1,...,N-1. \tag{3.8}$$

Suppose there is a nonzero solution ϕ to (3.7)-(3.8). We may assume that $\|\phi\|_{L^\infty}=1$. Since $w_L\to w(y)$ as $L\to\infty$, we conclude that $pw_L^{p-1}<\frac{1}{2}$ for $y\in B_R(0)\cap\Sigma_L$. Thus $|\phi(y)|\leq\max_{y\in B_R(0)}|\phi|e^{-\frac{1}{\sqrt{2}}(|y|-R)}$. So the maximum point of $|\phi|$ must occur in $B_{2R}(0)$. Letting $L\to+\infty$, we have that $\phi\to\phi_\infty$ which satisfies

$$\Delta\phi_\infty-\phi_\infty+pw^{p-1}\phi_\infty=0, \int_{\mathbb{R}^N}\phi_\infty\frac{\partial w}{\partial y_j}=0, j=1,...,N-1. \tag{3.9}$$

By Lemma 3.1, $\phi_\infty=c\frac{\partial w}{\partial y_N}=c\frac{w^{'}}{r}y_N$. We have seen that ϕ_∞ attains its maximum at some finite point. This is impossible since $\frac{\partial^2 w}{\partial^2 y_N}\neq 0$.

The proof of uniqueness of least energy solution when L is large is similar to that of nondegeneracy. In fact, suppose that there are two least energy solutions u_L and $u_L^{'}$ to (1.6). We may assume that both u_L and $u_L^{'}$ attain their maximum at $(0,1)$. Suppose that $u_L\not\equiv u_L^{'}$. Then letting $w_L(y):=L^{-\frac{2}{p-1}}u_L((0,1)+L^{-1}y), w_L^{'}(y):=L^{-\frac{2}{p-1}}u_L^{'}((0,1)+L^{-1}y)), \phi_L(y):=w_L(y)-w_L^{'}(y)$, we see that ϕ_L satisfies

$$\Delta\phi-\phi+V(y)\phi=0, \phi\in H^1(\Sigma_L) \tag{3.10}$$

where $V(y)=\frac{w_L^p-(w_L^{'})^p}{w_L-w_L^{'}}$. Since $V(y)\to pw^{p-1}(y)$ as $L\to+\infty$ and $\nabla\phi_L(0)=0$, the rest of the proof is exactly the same as before. We omit the details.

4. The critical exponent Case: the proof of (3) of Theorem 1.1

In this section, we assume that $p = \frac{N+2}{N-2}$. We consider two cases: L is small and L is large

4.1. Critical exponent case I: L small.

It is well-known (see [13]) that the solutions to the following problem

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \text{ in } \mathbb{R}^N, u > 0 \tag{4.1}$$

are given by

$$U_{\epsilon,a} = c_N\left(\frac{\epsilon}{\epsilon^2 + |x-a|^2}\right)^{\frac{N-2}{2}} \tag{4.2}$$

for some $\epsilon > 0$ and $a \in \mathbb{R}^N$.

Let

$$S = \frac{\int_{\mathbb{R}^N} |\nabla U_{1,0}|^2}{(\int_{\mathbb{R}^N} U_{1,0}^{\frac{2N}{N-2}})^{\frac{N-2}{N}}}, \ S_{\frac{1}{2}} = (\frac{1}{2})^{\frac{2}{N}} S. \tag{4.3}$$

We have the following lemma, whose proof follows from classical "concentration-compactness" principle of P.L. Lions [20], [21].

Lemma 4.1. *Let $p = \frac{N+2}{N-2}$. If*

$$c(L) < S_{\frac{1}{2}} \tag{4.4}$$

then $c(L)$ is attained.

As a Corollary, we have

Corollary 4.2. *Let $N \geq 4$. Then for L sufficiently small, $c(L)$ is attained.*

Proof. We just need to verify (4.4) for L small. Now we compute

$$\begin{aligned}
(4.5)\qquad & \int_\Sigma U_{\epsilon,0}^{\frac{2N}{N-2}} \\
&= c_N^{\frac{2N}{N-2}} \int_\Sigma \left(\frac{\epsilon}{\epsilon + |x|^2}\right)^N dx \\
&= c_N^{\frac{2N}{N-2}} \int_0^{\frac{1}{\epsilon}} \left(\int_{\mathbb{R}^{N-1}} \frac{1}{(1+t^2+|y'|^2)^N} dy' dt\right. \\
&= c_N^{\frac{2N}{N-2}} \int_0^{\frac{1}{\epsilon}} (1+t^2)^{-\frac{N+1}{2}} dt \left(\int_{\mathbb{R}^{N-1}} \frac{1}{(1+|y'|^2)^N} dy'\right. \\
(4.6)\qquad &= c_N^{\frac{2N}{N-2}} \left[\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^N} dy - \frac{\epsilon^N}{N} \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|y'|^2)^N} dy' + o(\epsilon^N)\right].
\end{aligned}$$

Similarly

$$\int_\Sigma |\nabla U_{\epsilon,0}|^2 = c_N^2(N-2)\int_{R_+^N} |\nabla U_{1,0}|^2 dy \tag{4.7}$$

$$-c_N^2\epsilon^{N-2}\int_{\mathbb{R}^{N-1}} \frac{1}{(1+|y'|^2)^{N-1}}dy' + c_N^2\frac{\epsilon^N(N-2)}{N}\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|y'|^2)^N}dy' + O(\epsilon^N).$$

On the other hand, for $N \geq 5$,

$$\int_\Sigma U_{\epsilon,0}^2 = c_N^2\int_\Sigma \left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{N-2} = c_N^2\epsilon^2\int_{\mathbb{R}_+^N}\frac{1}{(1+|y|^2)^{N-2}}dy \tag{4.8}$$

Thus for $N \geq 5$

$$c(L) \leq \frac{\int_\Sigma(|\nabla U_{\epsilon,0}|^2 + L^2U_{\epsilon,0}^2)}{(\int_\Sigma U_{\epsilon,0}^{p+1})^{\frac{N-2}{N}}} = \frac{A_0 - B_0\epsilon^{N-2} + L^2C_0\epsilon^2 + O(\epsilon^N)}{(D_0+O(\epsilon^N))^{\frac{N-2}{N}}} < \frac{A_0}{D_0^{\frac{2}{p+1}}} = S_{\frac{1}{2}}$$

if L is small and ϵ is small. Here

(4.9)

$$A_0 = \int_{\mathbb{R}_+^N}|\nabla U_{1,0}|^2, B_0 = c_N^2\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|y'|^2)^{N-1}}dy', C_0 = \int_{\mathbb{R}_+^N}U_{1,0}^2, D_0 = \int_{\mathbb{R}_+^N}U_{1,0}^{\frac{2N}{N-2}}$$

Applying Lemma (4.1), for L small, $c(L)$ is attained (possibly by a trivial solution).

For $N = 4$, we have

$$\int_\Sigma U_{\epsilon,0}^2 = c_N^2\epsilon^2\log\frac{1}{\epsilon}\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|y'|^2)^2}dy' \tag{4.10}$$

Similar arguments as before show that $c(L) < S_{\frac{1}{2}}$ for L small. □

Remark: In fact, Corollary 4.2 is also true for $N \geq 3$. Another proof is to use the inequality $c(L) \leq c^*(L)$.

4.2. **Critical exponent case II: L large.** The main theorem in this section is the following

Theorem 4.3. *For L large, $c(L)$ is not attained.*

We first assume that $N \geq 5$. Later on we will show how one can modify the arguments to deal with the case of $N = 4$.

We prove it by contradiction. Suppose that $c(L)$ is attained by some u_L for a sequence of $L = L_i \to +\infty$. Note that the computations of Corollary 4.2 show that

$$c(L) \leq S_{\frac{1}{2}}. \tag{4.11}$$

We claim that $c(L)$ is attained for all L. In fact, let $L < L_i$ for some L_i. Then we have

$$c(L) \leq \frac{\int_\Sigma (|\nabla u_{L_i}|^2 + L^2 u_{L_i}^2)}{(\int_\Sigma u_{L_i}^{p+1})^{\frac{2}{p+1}}} < c(L_i) \leq S_{\frac{1}{2}}. \tag{4.12}$$

By Lemma 4.1, $c(L)$ is attained.

By Steiner symmetrization, we may assume that u_L is symmetric in $x^{'}$ and increasing in x_N.

We rescale $\tilde{u}_L = L^{-\frac{2}{p-1}} u_L(L^{-1}y)$ and obtain that

$$c(L) = \frac{\int_{\Sigma_L} (|\nabla \tilde{u}_L|^2 + \tilde{u}_L^2)}{(\int_{\Sigma_L} \tilde{u}_L^{p+1})^{\frac{2}{p+1}}} \leq S_{\frac{1}{2}}$$

where $\Sigma_L = \mathbb{R}^{N-1} \times (0, L)$. This implies that

$$\int_{\Sigma_L} (|\nabla \tilde{u}_L|^2 + \tilde{u}_L^2) \leq C, \int_{\Sigma_L} \tilde{u}_L^{p+1} \leq C. \tag{4.13}$$

We claim that $\tilde{u}_L$ must blow up. If not, by taking a subsequence of L and extending $\tilde{u}_L$ to $\mathbb{R}^{N-1} \times (-L, L)$, we see that v_L converges to a positive solution of

$$\Delta v - v + v^{\frac{N+2}{N-2}} = 0, v \in H^1(\mathbb{R}^N)$$

which is impossible. In fact we have that

$$\frac{1}{\tilde{u}_L(0)} \tilde{u}_L((\tilde{u}_L(0))^{-\frac{p-1}{2}}(y) \to U_{1,0}(y)$$

in $C^2_{loc}(\mathbb{R}^N_+)$ as $L \to \infty$.

Let ϵ be such that

$$\tilde{u}_L(0) = \epsilon^{-\frac{N-2}{2}} \tag{4.14}$$

and set

$$v_\epsilon(y) = \epsilon^{-\frac{N-2}{2}} \tilde{u}_L(\epsilon y). \tag{4.15}$$

Then it is easy to see that v_ϵ satisfies

$$\Delta v_\epsilon - \epsilon^2 v_\epsilon + v_\epsilon^p = 0 \text{ in } \Sigma_{\frac{L}{\epsilon}}, \frac{\partial v_\epsilon}{\partial \nu} = 0 \text{ on } \Sigma_{\frac{L}{\epsilon}} \tag{4.16}$$

and that $v_\epsilon(y) \to U_{1,0}(y)$ in $C^2_{loc}(\mathbb{R}^N_+)$.

We now require the following crucial estimate.

Lemma 4.4.

$$v_\epsilon(y) \leq \frac{C}{(1 + |y|^2)^{\frac{N-2}{2}}}. \tag{4.17}$$

Proof. The derivation of this estimate follows exactly the proof of Theorem 2.1 of [14]. In fact, our situation is simpler as there is no need to straighten the boundary (see [14]). □

Let V_ϵ be the unique solution of the following linear problem

$$\Delta V_\epsilon - \epsilon^2 V_\epsilon + U_{1,0}^{\frac{N+2}{N-2}} = 0 \text{ in } \Sigma_{\frac{L}{\epsilon}}, \quad \frac{\partial V_\epsilon}{\partial \nu} = 0 \text{ on } \partial\Sigma_{\frac{L}{\epsilon}}. \tag{4.18}$$

Now we decompose

$$V_\epsilon = U_{1,0} - \varphi_\epsilon$$

Then we have

$$\begin{cases} \Delta\varphi_\epsilon - \epsilon^2\varphi_\epsilon + \epsilon^2 U_{1,0} = 0 \text{ in } \Sigma_{\frac{L}{\epsilon}}, \\ \frac{\partial\varphi_\epsilon}{\partial\nu} = \frac{\partial U_{1,0}}{\partial\nu} \text{ on } \partial\Sigma_{\frac{L}{\epsilon}}. \end{cases} \tag{4.19}$$

Let us set

$$\varphi_\epsilon = \epsilon^2\varphi_0(y) + \varphi_{\epsilon,1} \tag{4.20}$$

where φ_0 is the unique solution of the following problem

$$\Delta\varphi_0 + U_{1,0}(y) = 0 \text{ in } \mathbb{R}^N, \varphi_0(y) = \varphi_0(|y|), \varphi_0 \to 0 \text{ as } |y| \to +\infty \tag{4.21}$$

Note that since

$$U_{1,0}(y) \le \frac{C}{(1+|y|)^{N-2}}$$

where $N-2>2$, there exists a unique solution to (4.21).

Then $\varphi_{\epsilon,1}$ satisfies

$$\begin{cases} \Delta\varphi_{\epsilon,1} - \epsilon^2\varphi_{\epsilon,1} - \epsilon^4\varphi_0(y) = 0 \text{ in } \Sigma_{\frac{L}{\epsilon}}, \\ \frac{\partial\varphi_{\epsilon,1}}{\partial\nu} = \frac{\partial}{\partial\nu}[U_{1,0} - (\epsilon^2)\varphi_0] \text{ on } \partial\Sigma_{\frac{L}{\epsilon}}. \end{cases} \tag{4.22}$$

We claim that

Lemma 4.5.

$$|\varphi_{\epsilon,1}| \le C\epsilon^{N-2} + o(\epsilon^2)\frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}. \tag{4.23}$$

Now we let

$$v_\epsilon(y) = V_\epsilon(y) + \phi_\epsilon(y) \tag{4.24}$$

Then ϕ_ϵ satisfies

$$\begin{cases} \Delta\phi_\epsilon - \epsilon^2\phi_\epsilon + (V_\epsilon+\phi_\epsilon)^{\frac{N+2}{N-2}} - U_{1,0}^{\frac{N+2}{N-2}} = 0 \text{ in } \Sigma_{\frac{L}{\epsilon}}, \\ \frac{\partial\phi_\epsilon}{\partial\nu} = 0 \text{ on } \partial\Sigma_{\frac{L}{\epsilon}} \end{cases} \tag{4.25}$$

We also claim that

Lemma 4.6.

$$|\phi_\epsilon(y)| \le C\epsilon^2 \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}. \tag{4.26}$$

Postponing the proofs of Lemma (4.5) and Lemma (4.6) to the appendix, we can conclude that we have a contradiction to (4.11) by establishing the following:

$$c(L) > S_{\frac{1}{2}}. \tag{4.27}$$

First we note that

$$c(L) = \frac{\int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla v_\epsilon|^2 + \epsilon^2 v_\epsilon^2)}{\left(\int_{\Sigma_{\frac{L}{\epsilon}}} v_\epsilon^{p+1}\right)^{\frac{2}{p+1}}}. \tag{4.28}$$

Then we have

$$\begin{aligned}
\int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla v_\epsilon|^2 + \epsilon^2 v_\epsilon^2) &= \int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla(V_\epsilon + \phi_\epsilon)|^2 + \epsilon^2 (V_\epsilon + \phi_\epsilon)^2) \\
&= \int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla V_\epsilon|^2 + \epsilon^2 V_\epsilon^2) + 2\int_{\Sigma_{\frac{L}{\epsilon}}} (\nabla V_\epsilon \nabla \phi_\epsilon + \epsilon^2 V_\epsilon \phi_\epsilon) \\
&+ \int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla \phi_\epsilon|^2 + \epsilon^2 \phi_\epsilon^2) \\
&= I_1 + 2I_2 + I_3
\end{aligned}$$

where I_1, I_2 and I_3 are defined by the three terms at the last equality.

Quantity I_1 can be computed as follows:

$$I_1 = \int_{\Sigma_{\frac{L}{\epsilon}}} U_{1,0}^p V_\epsilon = \int_{\Sigma_{\frac{L}{\epsilon}}} U_{1,0}^p (U_{1,0} - \varphi_\epsilon)$$

$$= \int_{\mathbb{R}^N_+} U_{1,0}^{p+1} - \epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + o(\epsilon^2) \tag{4.29}$$

where we have used

$$\int_{\mathbb{R}^N_+} U_{1,0}^p \varphi_0 = \int_{\mathbb{R}^N_+} U_{1,0}^2 > 0. \tag{4.30}$$

For the quantity I_2, from the equation for V_ϵ, it follows that

$$I_2 = \int_{\Sigma_{L\epsilon}} U_{1,0}^p \phi_\epsilon = O(\epsilon^2). \tag{4.31}$$

By Lemma 4.5, we have

$$I_3 = \int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla(\phi_\epsilon)|^2 + \epsilon^2(\phi_\epsilon)^2) = o(\epsilon^2).$$

So

$$\int_{\Sigma_{\frac{L}{\epsilon}}} (|\nabla v_\epsilon|^2 + \epsilon^2 v_\epsilon^2) \geq \int_{\mathbb{R}^N_+} |\nabla U_{1,0}|^2 - \epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + 2I_2 + o(\epsilon^2). \tag{4.32}$$

Next, using Lemma 4.5 again, we obtain that

$$\begin{aligned}
\int_{\Sigma_{\frac{L}{\epsilon}}} v_\epsilon^{p+1} &= \int_{\Sigma_{\frac{L}{\epsilon}}} (V_\epsilon + \phi_\epsilon)^{p+1} \\
&= \int_{\Sigma_{\frac{L}{\epsilon}}} V_\epsilon^{p+1} + (p+1) \int_{\Sigma_{\frac{L}{\epsilon}}} V_\epsilon^p \phi_\epsilon + o(\epsilon^2) \\
&= \int_{\Sigma_{\frac{L}{\epsilon}}} V_\epsilon^{p+1} + (p+1) \int_{\Sigma_{\frac{L}{\epsilon}}} U_{1,0}^p \phi_\epsilon + o(\epsilon^2) \\
&= \int_{\mathbb{R}^N_+} U_{1,0}^{p+1} - (p+1)\epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + I_2 + o(\epsilon^2). \qquad (4.33)
\end{aligned}$$

Combining (4.32) and (4.33), we obtain that

$$\begin{aligned}
c(L) &\geq \frac{\int_{\mathbb{R}^N_+} |\nabla U_{1,0}|^2 - \epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + I_2}{(\int_{\mathbb{R}^N_+} U_{1,0}^{p+1} - (p+1)\epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + (p+1)I_2 + o(\epsilon^2))^{\frac{2}{p+1}}} \qquad (4.34) \\
&\geq S_{\frac{1}{2}} + \epsilon^2 \int_{\mathbb{R}^N_+} U_{1,0}^2 + o(\epsilon^2) + O(|I_2|^2) > S_{\frac{1}{2}}
\end{aligned}$$

which proves (4.27). □

Finally, when $N = 4$, we have to replace $\varphi_0(y)$ be the following function

$$\varphi_0(y) = \log \frac{1}{1+|y|^2} + (\Delta)^{-1}\left(\frac{1}{(1+|y|^2)^2}\right)$$

and ϵ^2 by $\epsilon^2 \log \frac{1}{\epsilon}$. The rest of the proof is unchanged.

4.3. **Completion of proof of (3) of Theorem 1.1.** Let $p = \frac{N+2}{N-2}$ and $N \geq 4$. By Corollary 4.2, $c(L)$ is attained for L small. Set

$$L_{**} = \sup\{L | c(L) \text{ is attained for } l \in (0, L)\}. \tag{4.35}$$

By Corollary 4.2 and Theorem 4.3, $0 < L_{**} < +\infty$.

We claim that $c(L)$ is also attained at $L = L_{**}$. In fact, if not, let $u_L, L < L_{**}$ be the minimizers of $c(L)$. Then as $L \to L_{**}$, u_L must blow up. But similar arguments as in Theorem 4.3 shows that this is impossible.

Now we show that for $L > L_{**}$, $c(L)$ is not attained. In fact, suppose $c(L)$ is attained for some $L_0 > L_{**}$. then certainly, $c(L_0) \le S_{\frac{1}{2}}$. Let the minimizer of $c(L_0)$ be u_{L_0}. Then

$$c(L) < c(L_0) \le S_{\frac{1}{2}}, \text{ for } L < L_0. \tag{4.36}$$

By Lemma 4.1, $c(L)$ is attained if $L < L_0$, which is a contradiction with the definition of L_{**}.

Finally for L small, we have $c(L) < c(\frac{L_{**}}{2}) \le S_{\frac{1}{2}}$. By the same analysis as in Lemma 2.1 the minimizer is uniformly bounded. Hence for L small $c(L)$ is archived by trivial solution. Now similar proofs as in Lemma 2.4 yield another constant $L_* < L_{**}$ such that for $L \le L_*$, $c(L)$ is achieved by trivial solution and for $L \in (L_*, L_{**}]$, $c(L)$ is attained by a nontrivial constant.

Thus, (3) of Theorem 1.1 is proved. □

Appendix A: Proof of Lemma 4.5 and Lemma 4.6.

To prove Lemma 4.5 and 4.6, we introduce two weighted L^∞ spaces. For f a function in $\Sigma_{\frac{L}{\epsilon}}$, we define the following weighted L^∞-norms

$$\|f\|_* = \sup_{x \in \Sigma_{\frac{L}{\epsilon}}} \left| (1+|y|^2)^{\frac{N-2}{2}} f(x) \right|$$

and

$$\|f\|_{**} = \sup_{x \in \Sigma_{\frac{L}{\epsilon}}} \left| (1+|y|^2)^{\frac{N-1}{2}} f(x) \right|.$$

Lemma A. *Let $f \in L^\infty(\Sigma_{\frac{L}{\epsilon}})$ be such that*

$$\|f\|_{**} < +\infty$$

and u satisfy

$$-\Delta u + \epsilon^2 u = f \quad in\ \Sigma_{\frac{L}{\epsilon}},\ \frac{\partial u}{\partial \nu} = 0 \ on\ \partial \Sigma_{\frac{L}{\epsilon}}.$$

Then we have

$$|u(x)| \le \int_{\Sigma_{\frac{L}{\epsilon}}} \frac{C}{|x-y|^{N-2}} |f(y)| dy, \tag{4.37}$$

where C is independent of $L \ge L_0$. As a consequence, we have

$$\|u\|_* \le C \|f\|_{**}. \tag{4.38}$$

To prove Lemma 4.5, we decompose $\varphi_{\epsilon,1}$ into two parts:

$$\varphi_{\epsilon,1} = \varphi_{\epsilon,1}^1 + \varphi_{\epsilon,1}^2$$

where $\varphi_{\epsilon,1}$ satisfies

$$\Delta\varphi^1_{\epsilon,1}-\epsilon^2\varphi^1_{\epsilon,1}=0 \text{ in } \Sigma_{\frac{L}{\epsilon}}, \frac{\partial\varphi^1_{\epsilon,1}}{\partial\nu}=\frac{\partial}{\partial\nu}(U_{1,0}-\epsilon^2\varphi_0(y)) \text{ on } \partial\Sigma_{\frac{L}{\epsilon}},$$

and $\varphi^2_{\epsilon,1}$ satisfies

$$\Delta\varphi^2_{\epsilon,1}-\epsilon^2\varphi^2_{\epsilon,1}-\epsilon^2\varphi_0(y)=0 \text{ in } \Sigma_L, \frac{\partial\varphi^2_{\epsilon,1}}{\partial\nu}=0 \text{ on } \partial\Sigma_{\frac{L}{\epsilon}}$$

The first part can be estimated by asymptotic analysis while the second part follows from comparison principle.

To prove Lemma 4.6, we note that ϕ_ϵ satisfies

$$\Delta\phi_\epsilon-\epsilon^2\phi_\epsilon+pV_\epsilon^{p-1}\phi_\epsilon+E_\epsilon+N_\epsilon[\phi_\epsilon]=0$$

where

$$E_\epsilon=V_\epsilon^p-U_{1,0}^p,$$
$$N_\epsilon[\phi_\epsilon]=(V_\epsilon+\phi_\epsilon)^p-V_\epsilon^p-pV_\epsilon^{p-1}\phi_\epsilon.$$

We argue by contradiction. Let $\tilde{\phi}_\epsilon=\frac{\phi_\epsilon}{\epsilon^2}$. We assume that

$$\|\tilde{\phi}_\epsilon\|_*\to+\infty. \tag{4.39}$$

Set

$$\Phi_\epsilon=\frac{\tilde{\phi}_\epsilon}{\|\tilde{\phi}_\epsilon\|_*}. \tag{4.40}$$

Then it is easy to see that Φ_ϵ satisfies

$$\Delta\Phi_\epsilon-\epsilon^2\Phi_\epsilon+pV_\epsilon^{p-1}\Phi_\epsilon+(\|\tilde{\phi}_\epsilon\|_*)^{-1}\epsilon^{-2}E_\epsilon+(\|\tilde{\phi}_\epsilon\|_*)^{-1}\epsilon^{-2}N_\epsilon=0.$$

As $\epsilon\to0$, $\Phi_\epsilon\to\Phi_0$ where Φ_0 satisfies

$$\Delta\phi_0+pU_{1,0}^{p-1}\phi_0=0 \text{ in } \mathbb{R}^N.$$

It is well-known (see [26]) that $\Phi_0=a_0\frac{\partial U_{\lambda,0}}{\partial\lambda}|_{\lambda=1}+\sum_{j=1}^{N-1}a_j\frac{\partial U_{1,0}}{\partial y_j}$ for some constants $a_j, j=0,1,...,N-1$.

Now since both v_ϵ and V_ϵ are symmetric in x', we see that $\frac{\partial}{\partial y_j}\Phi_\epsilon(0)=0, j=1,...,N-1$. We also have that $\Phi_\epsilon(0)=\frac{v_\epsilon(0)-V_\epsilon(0)}{\epsilon^2\|\tilde{\phi}_\epsilon\|_*}=o(1)$ and hence $\Phi_0(0)=0$. This together with $\frac{\partial}{\partial y_j}\phi_0(0)=0$ will force $a_j=0, j=0,1,...,N-1$ and hence $\Phi_0=0$.

On the other hand, from the equation for Φ_ϵ, we have that

$$\|V_\epsilon^{p-1}\Phi_\epsilon\|_{**}=o(1), \|\epsilon^{-2}E_\epsilon\|_{**}=O(1),$$

and by Lemma A, we then arrive at

$$\|\Phi_\epsilon\|_*=o(1)$$

A contradiction to (4.40)!

So Lemma 4.5 is proved. □

It remains to prove Lemma A. By a scaling, we may assume that $\epsilon = 1$. Then

$$u(x) = \int_{\Sigma_L} G_L(x,y) f(y) dy$$

where $G_L(x,y)$ is the Green's function

$$\Delta G_L - G_L + \delta_\xi = 0 \text{ in } \Sigma_L, \frac{\partial G_L}{\partial \nu} = 0 \text{ on } \partial \Sigma_L.$$

We have to show that

$$G_L(x,y) \leq \frac{C}{|x-y|^{N-2}} \tag{4.41}$$

where C is independent of $L \geq L_0$, But (4.41) follows from standard potential estimates. See [27]. Note also that $|f(y)| \leq \frac{C}{(1+|y|)^{N-1}} < \frac{C}{(1+|y|)^3}$ so the integral $\int_{\Sigma_\epsilon} \frac{|f(y)|}{|x-y|^{N-2}} \leq C\|f\|_{**} \frac{1}{(1+|y|)^{N-2}}$.

Appendix B: Asymptotic behavior when $L \to L^*$

In this appendix, we study the asymptotic behavior of the least energy solution when $L \to L^*$. Let $L = L_*$ and w_0 be the unique radial solution of (2.7). It is clear that when $L \to L_*$, $u_L \to w_0$ uniformly. In the following, we shall derive the next two order terms in the expansion of u_L.

First, we consider the following linear problem

$$\Delta \phi - L_*^2 \phi + p w_0^{p-1} \phi = 0 \text{ in } \Sigma, \phi \in H^1(\Sigma), \ \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Sigma \tag{4.42}$$

Then by separation of variables we have

$$\phi = \sum_{j=1}^{N-1} \frac{\partial w_0}{\partial y_j} + c_N \Phi_0 \tag{4.43}$$

where

$$\Phi_0 = \phi_0(|y^{'}|) \cos(\pi y) \tag{4.44}$$

and ϕ_0 is the principal eigenfunction of $\Delta_{y'} - L_*^2 + p w_0^{p-1}$. (We choose Φ_0 so that $\int_\Sigma \Phi_0^2 = 1$.)

As a consequence of the Fredholm Alternative, there exists a unique solution to the following problem

$$\Delta \phi - L_*^2 \phi + p w_0^{p-1} \phi = f \text{ in } \Sigma, \phi \in H^1(\Sigma), \ \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Sigma \tag{4.45}$$

with

$$\int_\Sigma \phi \frac{\partial w_0}{\partial y_j} = \int_\Sigma \phi \Phi_0 = \int_\Sigma f \frac{\partial w_0}{\partial y_j} = \int_\Sigma f \Phi_0 = 0 \tag{4.46}$$

such that

$$\|\phi\|_{H^2(\Sigma)} \le C\|f\|_{L^2(\Sigma)} \tag{4.47}$$

Let us denote $L_0 := \Delta - L_*^2 + pw_0^{p-1}$ and $\mathcal{K}_0 = \text{span}\ \{\frac{\partial w_0}{\partial y_j}, \Phi_0\}$. Now set

$$\delta = \int_\Sigma (u_L - w_0)\Phi_0 \tag{4.48}$$

We now claim that

$$u_L = w_0 + \delta\Phi_0 + \delta^2\Phi_1 + \delta^3\Phi_2 \tag{4.49}$$

where $\|\Phi_2\|_{H^2} \le C$ and Φ_1 satisfies

(4.50)

$$L_0\Phi_1 + \frac{L_*^2 - L^2}{\delta^2} w_0 + \frac{p(p-1)}{2} w^{p-2}\Phi_0^2 = 0 \text{ in } \Sigma, \Phi_1 \perp \mathcal{K}_0,\ \frac{\partial \Phi_1}{\partial \nu} = 0 \text{ on } \Sigma.$$

We prove it by expansion. Let $u_L = w_0 + \delta\Phi_0 + \Phi_{1,L}$. Then $\Phi_{1,L} \perp \mathcal{K}_0$ and $\Phi_{1,L} = o(1)$. Since $\Delta(w_0 + \delta\Phi_0) - L^2(w_0 + \delta\Phi_0) + (w_0 + \delta\Phi_0)^p = (L_*^2 - L^2)w_0 + \delta^2 \frac{p(p-1)}{2} w^{p-2}\Phi_0^2 + \mathcal{O}(|\delta|^3 + |L_*^2 - L^2||\delta|)$. We conclude that $\Phi_{1,L} = \mathcal{O}(|L_*^2 - L^2| + \delta^2)$. Now we let Φ_1 be as defined in (4.50). Decomposing u_l as in (4.49), we see that Φ_2 satisfies

$$L_0[\Phi_2] + \frac{L_*^2 - L^2}{\delta^2}\Phi_0 + p(p-1)w^{p-2}\Phi_0\Phi_1 + \frac{p(p-1)(p-2)}{6} w^{p-3}\Phi_0^3 + O(\delta) = 0$$

and hence we have that

$$\frac{L_*^2 - L^2}{\delta^2} \int_\Sigma \Phi_0^2 + p(p-1)w^{p-2} \int_\Sigma \Phi_0^2\Phi_1 + \frac{p(p-1)(p-2)}{6} \int_\Sigma w^{p-3}\Phi_0^4 = 0$$

which gives the desired precise formula for δ in terms of $L_*^2 - L^2$.

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